Exercises's solutions - Probability (2/2)

Exo 1

In a class of 80 students, the professor calls on 1 student chosen at random for a recitation in each class period. There are 32 class periods in a term.

1. Let X be the random variable which represents the number of times that a given student is chosen in a term. This is a repetition of 32 Bernoulli process (either the student is chosen, either he is not) and X follows a Binomial distribution: $X\tilde{b}(n,p)$ with $n = 32$, $p = \frac{1}{80}$.

$$P(X = k) = \binom{32}{k} (\frac{1}{80})^k (\frac{79}{80})^{n-k}$$

2. A Poisson approximation is

$$P(X = K)_{\lambda=\frac{32}{80}} = \frac{1}{k!}\left(\frac{32}{80}\right)^k e^{-\lambda}$$

We use this formula to estimate the probability that a given student is called upon more than twice.

$$\begin{aligned} P(X > 2) &= 1 - P(X \leq 2) \\ &= 1 - P(X = 0) - P(X = 1) - P(X = 2) \\ &= 1 - e^{-\frac{32}{80}} - \frac{32}{80}e^{\frac{32}{80}} - \frac{1}{2}\left(\frac{32}{80}\right)^2 e^{-\frac{32}{80}} \\ &\approx 0.8\% \end{aligned}$$

Exo 2

Reese Prosser never puts money in a 10-cent parking meter in Hanover. He assumes that there is a probability of .05 that he will be caught. The first offense costs nothing, the second costs 2 dollars, and subsequent offenses cost 5 dollars each.

1. Let's compute the expected cost of frauding. Let X be the number of times Reese is caught. X follows a binomial distribution with $p = 0.05$ and $n = 100$. It can be approximate with a Poisson process with $\lambda = 5$.

$$\begin{aligned} E(X) = &P(X = 0) * 0 \\ &+ P(X = 1) * 0 \\ &+ P(X = 2) * 2 \\ &+ P(X = 3) * (2 + 5 * 1) \\ &+ \ldots \\ &+ P(X = 100) * (2 + 5 * 98) \\ &\approx 17.2\$ \end{aligned}$$

Reese would be better off paying the the meter each time.

Exo 3
A basket contains eight white balls and two black balls. We draw three balls from the basket without replacement. Let X denote the number of white balls drawn.

1. X follows a hypergeometric distribution with parameter $N = 10$, $n = 3$ and $p = \frac{8}{10}$

2.

$$\begin{aligned} E(X) &= n * p \\ &= \frac{21}{10} \\ V(X) &= \frac{np(1-p)(N-n)}{N-1} \\ &\approx 0.373 \end{aligned}$$

Exo 4
A restaurant has three menus: A,B and C. Each customer chooses one and only menu among the three menus. We assume that each customer made a random, independent choice. Let A_n, B_n and C_n be the number of customers amid the n customers that showed up this day, that choose menu A, B, C respectively. For instance, A_3 denotes the number of people who have chosen menu A among the 3 customers that showed up this day.

1. Let A_2 be the number of clients who have chosen the menu A among the two clients that went eating in the restaurant on a given day. The choice of each client is independent and choosing menu A has a probability of $\frac{1}{3}$, $A_2 \sim \mathcal{B}(n = 2, p = \frac{1}{3})$. We have:

$$\begin{aligned} E(X) &= \frac{2}{3} \\ V(X) &= \frac{4}{9} \end{aligned}$$

2. $A_n \sim \mathcal{B}(n, \frac{1}{3})$

3. $n - A_n$ represents the number of people who have not chosen menu A. Not choosing menu A is a bernoulli process with $p = \frac{2}{3}$. Thus:

$$n - A_n \sim (n, p = \frac{2}{3})$$

4. Each client choose the same menu if each client choose menu A or each client choose menu B or each client choose menu C. Denote T the event "each client choose the same

menu". We have :

$$\begin{aligned}P(T) &= P((A_n = n \cup (B_n = n) \cup (C_n = n)) \\ &= P(A_n = n) \cup P(B_n = n) \cup P(C_n = n) \\ &= 3 * P(A_n = n) \\ &= 3 \binom{n}{n} \frac{1}{3^n} \left(\frac{2}{3}\right)^0 \\ &= \frac{1}{3^{n-1}}\end{aligned}$$

5. Let U be the event "each menu is chosen at least once". We have:

$$P(U) = P((A_n \geq 1) \cap (B_n \geq 1)(C_n \geq 1))$$

We look for the probability of the complementary event, $P(\bar{U})$: "one menu is never chosen"

$$\begin{aligned}P(\bar{U}) &= P((A_n = 0) \cup (B_n = 0) \cup (C_n = 0)) \\ &= P(A_n = 0) + P(B_n = 0) + P(C_n = 0) \\ &- [P((A_n = 0) \cap (B_n = 0)) + P((A_n = 0) \cap (C_n = 0)) + P((B_n = 0) \cap (C_n = 0))] \\ &+ P((A_n = 0) \cap (B_n = 0) \cap (C_n = 0)\end{aligned}$$

We know that

$$\begin{aligned}P(A_n = 0) = P(B_n = 0) = P(C_n = 0) &= \left(\frac{2}{3}\right)^n \\ P((A_n = 0) \cap (B_n = 0) \cap (C_n = 0)^1 &= 0 \\ P((A_n = 0) \cap (B_n = 0)) = P(C_n = n) &= \frac{1}{3^n}\end{aligned}$$

Therefore:

$$\begin{aligned}P(U) &= 1 - 3 * \left(\frac{2}{3}\right)^n + 3 * \frac{1}{3^n} \\ &= \frac{3^{n-1} - 2^n - 1}{3^{n-1}}\end{aligned}$$

Exo 5

A royal family has children until it has a boy or until it has three children, whichever comes first. Assume that each child is a boy with probability 1/2.

1. There are three possible outcomes:

    - 1 boy and 0 girl, with probability $\frac{1}{2}$

[1] it is impossible that no menu has been selected.

- 1 girl and then 1 boy with probability $\frac{1}{4}$
- 2 girls and then 1 boy with probability $\frac{1}{8}$
- 3 girls with probability $\frac{1}{8}$

The expected number of boy and girls is therefore:

$$\begin{aligned}\mathbb{E}_{boy} &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \\ &= \frac{7}{8} \\ \mathbb{E}_{girl} &= \frac{1}{4} + \frac{2}{8} + \frac{3}{8} \\ &= \frac{7}{8}\end{aligned}$$

Exo 6

A basket contains r red balls ($r \geq 2$) and five black balls. The game consists of randomly selecting two balls from the basket at the same time.

- If both balls are red, we win 10 euros.
- If one of the two balls is red, we win 2 euros.
- If neither ball is red, we lost 6 euros.

1. Let X be the random variable denoting the number of red balls drawn from the basket. X follows a hypergeometric distribution with $N = 5 + r$, $p = \frac{r}{r+5}$ and $n = 2$. $X \sim \mathcal{H}(5 + r, 2, \frac{r}{r+5})$.
   Let p_r the probability to earn money:

$$\begin{aligned}p_r = P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - \frac{\binom{r}{0}\binom{5}{2}}{\binom{r+5}{2}} \\ &= \frac{r(r+9)}{(r+5)(r+4)}\end{aligned}$$

   Let G_r be the winnigs pocketed at the end of the game (depends of r).

2. The set of possible outcomes for G_r is $G_r(\Omega) = \{-6; +2; +10\}$. We have

$$\begin{aligned}P(G_r = -6) = P(X = 0) &= \frac{20}{(r+5)(r+4)} \\ P(G_r = +2) = P(X = 1) &= \frac{10r}{(r+5)(r+4)} \\ P(G_r = +10) = P(X = 2) &= \frac{r(r-1)}{(r+5)(r+4)}\end{aligned}$$

3. The expected value of G_r is :

$$\begin{aligned}\mathbb{E}(G_r) &= -6 * P(G_r = -6) + 2 * P(G_r = 2) + 10 * P(G_r = 10) \\ &= \frac{10r^2 + 10r - 120}{(r+5)(r+4)}\end{aligned}$$

4. The game is fair if $\mathbb{E}(G_r) = 0$ and $\mathbb{E}(G_r) = 0 \Leftrightarrow 10r^2 + 10r - 120 = 0$. This is a quadratic equation. We compute the discriminat of the equation.

   Note: For a quadratic equation $ax^2 + bx + c = 0$, the discriminant, denoted Δ, is $\Delta = b^2 - 4ac$. If the discriminant is positive, the equation has two disctincts roots (or solutions):

$$\frac{-b + \sqrt{\Delta}}{2a}$$
$$\frac{-b - \sqrt{\Delta}}{2a}$$

   If the discriminant is nil, the equation has one root: $-\frac{b}{2a}$ and if the discriminant is negative, the equation has no root in $\mathbb{R}$.

   Here $\Delta = 4900$ and the two roots of the equation are $r = 3$ and $r = -4$. Only $r = 3$ is possible and we conclude that the game is fair if there are three red balls.

Exo 7
We use the fact for a normal distribution, $\mathcal{N}(\mu, \sigma)$:

$$P(\mu - \sigma < X \leq \mu + \sigma) \approx 0.68$$
$$P(\mu - 2\sigma < X \leq \mu + 2\sigma) \approx 0.95$$

1. 0.025
2. 0.16
3. 0.68

Exo 8

1. 0.82
2. 0.16
3. 0.02

Exo 9
A typesetter makes, on the average, one mistake per 1000 words. Assume that he is setting a book with 100 words to a page. Let S_{100} be the number of mistakes that he makes on a single page.

1. This the repetition of 100 bernoulli process with $p = \frac{1}{1000}$. Thus, $S_{100} \sim \mathcal{B}(n = 100, p = \frac{1}{1000})$
2. $\mathcal{P}(\lambda = 0.1)$

Exo 10
Suppose that in a certain fixed amount A of blood, the average human has 40 white blood cells. Let X be the random variable which gives the number of white blood cells in a random sample of size A from a random individual.

1. We use the fact $P(X) \sim \mathcal{P}(\lambda = 40)$:

$$\begin{aligned} P(X \leq 38) \cup P(42 \leq X) &= 1 - P(X = 39) - P(X = 40) - P(X = 41) \\ &= 0.81 \end{aligned}$$

Exo 11
Give the appropriate distribution for each of the following random variables:

1. $X \sim \mathcal{U}(\Omega = \{1, 2, 3, 4, 5, 6\})$.
2. $X \sim \mathcal{B}(n = 3, p = \frac{1}{2})$.
3. $X \sim \mathcal{U}(\Omega = \{0, 00, 1, 2, \ldots, 36\})$.
4. $X \sim \mathcal{U}(\Omega = \{1, 2, \ldots, 365\})$.