Quantitative Tools - Level 1 - Fall 2015
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Lecture 3

Indexes

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1 Definitions

1.1 Elementary Index

Index Broadly, an index is a statistical measure that summarises and ranks specific observations. For example, the Human Development Index (HDI), the Big Mac Index or stockmarket index like the Dow Jones. More specifically, in Statistics, we are interest of a particular kind of index reflecting changes of one or several variables, called **index number**.
There are two kind of indexes :

- Elementary indexes, when the index is constructed on a single variable.
- Synthetic indexes, when the index aggregates several variables.

Definition 1.1. Index Number[1]
The index of a number is the ratio between the value of the number the current period and its value at the baseline period. It measures the relative variation of the value between the baseline period and the current period. Often, the ratio is multiplied by 100 and we say "index base 100 at such-and-such a period".
Indices are used for easy calculation and comparison of the trends in several numbers between two given periods.

Formally : Let X be the variable we want to construct an index between t_0 (the baseline period) and t_1 called I_{t_1/t_0}, then :

$$I_{t_1/t_0} = \frac{X_{t_1}}{X_{t_0}}$$

If we set the value of the index at 100 in t_0, then :

$$I_{t_1/t_0} = \frac{X_{t_1}}{X_{t_0}} * 100$$

Example 1.1. The Baguette Index
We want to construct the Index of the price of a 250g French baguette. We choose 2005 as the baseline period and we see how the price has changed between 2005 and 2014 (see Table 1 below). For each year N, we construct the index $I_{N/2005}$ as :

$$I_{N/2005} = \frac{\text{Price of a baguette in year } N}{\text{Price of a baguette in year 2005}} * 100$$

[1] as defined by the INSEE, the French National Institute of Statistics

Years	Price of a baguette (€ /kg)	Index
2005	€3.00	100
2006	€3.07	102.33
2007	€3.18	106.00
2008	€3.32	110.67
2009	€3.34	111.33
2010	€3.35	111.67
2011	€3.42	114.00
2012	€3.46	115.33
2013	€3.47	115.67
2014	€3.48	116.00

Table 1: Index of the price of a baguette in French Metropolitan area between 2005 and 2014

Based on the table, one can say that the index has increased by 11.67% between 2005 and 2010.

Definition 1.2. Change of an index
For comparing the change of the variable X between two periods t_j and t_i, with I_{t_j/t_0} the index of variable X between t_j and baseline period t_0, one has to use the classic formula of relative change :

$$\frac{\Delta X}{X} = \frac{I_{t_j/t_0} - I_{t_i/t_0}}{I_{t_i/t_0}}$$

Example 1.2. The Baguette Index
We want to know the relative change of the price of a baguette between 2008 and 2010 using the Baguette Index. We do :

$$\frac{I_{2010/2005} - I_{2008/2005}}{I_{2008/2005}} = 0.90\%$$

We conclude that the price of a baguette has increased by 0.9% between 2008 and 2010. We can verify the result by directly calculate the relative change using prices (see Lecture 1 notes for the formula).

Example 1.3. Example of elementary indexes on prices, quantities and aggregates values
We have the quantities and prices of different goods consumed in the Tunisian small village Tataouine

Years	Squid		Sweet pepper	
	Price	Quantities	Price	Quantities
Year 1	4	90	1	270
Year 2	7	60	2	240
Year 3	8	55	4	247.5

Table 2: Price & Quantities of goods consumed in Tataouine

We can construct elementary indexes of prices, quantities and **aggregate values** (or global values equals to price*quantity). Here is the table of the three elementary indexes for squid.

Years	Squid		
	Index of Prices	Index of Quantities	Index of aggregate values
Year 1	100	100	100
Year 2	175	66.7	116.7
Year 3	200	61.1	122.2

Table 3: Index of Price & Quantities & aggregate values of Squid

As you can see in the example, the index of aggregate values equals the product of the index of price and the index of quantities divided by 100, but also equals the index directly constructed on aggregate values (*i.e.* by considering the change of aggregate values).

1.2 Synthetic indexes

A synthetic index is an index that captures the change of a set of multiple variables. Inside the group of synthetic index, one can distinguish between unweighted and weighted indexes.

- An unweighted synthetic index is the arithmetic mean of the indexes of the variables forming to the set we are considering

- A weighted synthetic index is a weighted mean of the indexes of the variables forming the set.

When building a synthetic index that aggregates the changes of several values, the statistician/economist faces the following problem : usually both prices and quantities have changed. See the following example :

Example 1.4. Building a cost of living index in Tataouine
For instance, if we want to construct a price index of the goods consumed in Tataouine, that is a cost of living index in Tataouine. Let say that in year 1, a typical Tataouinian consumes 90 squids at $4 each and 270 sweet peppers at $1 each (since the typical Tataouinian meal is one squid cooked with three sweet peppers). But in year 2, prices and pattern of consumption change and a typical Tataouinian consumes now 60 squids at $7 each and 240 sweet peppers at $2 each.

We want to construct an index of the cost of living of a typical Tataouinian to see how the cost of living has changed between year 1 and year 2. One way to do it, would be to construct a price index for both the price of squids and sweet peppers and to choose the arithmetic mean of the two indexes (see Table 4). That is :

$$I_{2/1}^{\text{cost of living}} = \frac{1}{2} * I_{2/1}^{squid} + \frac{1}{2} * I_{2/1}^{\text{sweet pepper}}$$

	Index of Squid's prices	Index of Sweet Peppers' Price	Unweighted Mean Index
Year 1	100	100	100
Year 2	175	200	187.5

Table 4: Price Indexes of Squids and Sweet Peppers

We would conclude that the cost of living a typical Tatouinian has increased by 87.5% between year 1 and year 2. But one could object that a typical Tatouinian consumes for $360 of squids a year and only for $270 of sweet peppers. So that an increase in squid's price has a more severe effect on the change of his cost of living. So, you could choose to weight each price index in the aggregate index by the weight of the item in a typical Tataouinian pattern of consumption. That is :

$$I_{2/1}^{\text{cost of living}} = \frac{360}{360+270} * I_{2/1}^{squid} + \frac{270}{360+270} * I_{2/1}^{\text{sweet pepper}}$$

But, one could again object that the pattern of consumption has changed between year 1 and year 2. In year 2, a typical Tatouinian consumes for $420 of squids a year and for $480 of sweet peppers. So why would we prefer to choose year 1 rather than year 2 as the baseline for weights ?

The previous example shows that for building a price index, one has to *neutralise* the change of quantities and that for doing so, one has to choose a method for fixing them (usually a baseline period). For building an index of quantities (like the real growth of GDP), one would faces the opposite "puzzle" : neutralise the prices changes while depicting the changes of real GDP. We focus here on the two most common methods : the **Laspeyres index** (named after Etienne Laspeyres) and the **Paasche index** (named after Hermann Paasche)[2].

Definition 1.3. the Laspeyres index
When the value of the neutralised variable is set at the baseline period, the index is called a **Laspeyres index**. For a price index, a Laspeyres index is the ratio of the current price of the old basket of goods by the price at the baseline period of the old basket of goods.

Formally, the Laspeyres price index L between t_0 (baseline period) and t_1 called L_{t_1/t_0} of a basket of n goods is :

$$L_{t_1/t_0} = \frac{\sum_{i=1}^{n} p_{t_1}^i * q_{t_0}^i}{\sum_{i=1}^{n} p_{t_0}^i * q_{t_0}^i} \text{ where } \begin{cases} p_{t_1}^i & \text{is the price of the } i^{th} \text{ good at period } t_1 \\ q_{t_0}^i & \text{is the quantity of the } i^{th} \text{ good at period } t_0 \end{cases}$$

[2]Here, the Laspeyres and Paasche index are displayed with the value 1 at the baseline period t_0. The formula for the indexes in base 100 is the same with a $*100$ at the end

The Laspeyres index is the arithmetic mean of the elementary indexes weighted by the values at the baseline period. Indeed, let $E^i_{t_1/t_0} = \frac{p^i_{t_1}}{p^i_{t_0}}$ be the elementary index of good i. Then :

$$L_{t_1/t_0} = \frac{\sum_{i=1}^n q^i_{t_0} * p^i_{t_0} * \dfrac{p^i_{t_1}}{p^i_{t_0}}}{\sum_{i=1}^n p^i_{t_0} * q^i_{t_0}}$$

$$L_{t_1/t_0} = \sum_{i=1}^n \left(\frac{q^i_{t_0} * p^i_{t_0}}{\sum_{i=1}^n p^i_{t_0} * q^i_{t_0}} \right) E^i_{t_1/t_0}$$

$$\rightarrow L_{t_1/t_0} = \sum_{i=1}^n \alpha^i * E^i_{t_1/t_0} \qquad \text{where } \alpha^i = \frac{q^i_{t_0} * p^i_{t_0}}{\sum_{j=1}^n p^j_{t_0} * q^j_{t_0}}$$

The α^i are the share of good i in the total spending at the baseline period[3]. Thus, the Laspeyres index L_{t_1/t_0} is the weighted average (or arithmetic mean) of the elementary indexes of the n goods weighted by their share in the total spending at the baseline period.

Definition 1.4. the Paasche index
When the value of the neutralised variable is set at the current period, the index is called a **Paasche index**. For a price index, a Paasche index is the ratio of the current price of the current basket of goods by the price at the baseline period of the current basket of goods.

Formally, the Paasche price index P between t_0 (baseline period) and t_1 called P_{t_1/t_0} of a basket of n goods is :

$$P_{t_1/t_0} = \frac{\sum_{i=1}^n p^i_{t_1} * q^i_{t_1}}{\sum_{i=1}^n p^i_{t_0} * q^i_{t_1}} \text{ where} \begin{cases} p^i_{t_0} & \text{is the price of the } i^{th} \text{ good at period } t_0 \\ q^i_{t_1} & \text{is the quantity of the } i^{th} \text{ good at period } t_1 \end{cases}$$

The Paasche index is the harmonic mean of the elementary indexes weighted by the values at the current period. Indeed :

$$P_{t_1/t_0} = \frac{\sum_{i=1}^n p^i_{t_1} * q^i_{t_1}}{\sum_{i=1}^n q^i_{t_1} * p^i_{t_1} * \dfrac{p^i_{t_0}}{p^i_{t_1}}}$$

$$P_{t_1/t_0} = \frac{1}{\sum_{i=1}^n \left(\dfrac{q^i_{t_1} * p^i_{t_1}}{\sum_{i=1}^n p^i_{t_1} * q^i_{t_1}} \right) * \dfrac{p^i_{t_0}}{p^i_{t_1}}}$$

$$\rightarrow P_{t_1/t_0} = \frac{1}{\sum_{i=1}^n \beta^i * \dfrac{1}{E^i_{t_1/t_0}}} \qquad \text{where } \beta^i = \frac{q^i_{t_1} * p^i_{t_1}}{\sum_{j=1}^n p^j_{t_1} * q^j_{t_1}}$$

The β^i are the share of good i in the total spending at the current period. Thus, the Paasche index is the weighted harmonic mean of the elementary indexes of the n goods weighted by their share in the total spending at the current period.

Example 1.5. Laspeyres and Paasche price indexes in Tataouine
We want to calculate the Laspeyres L_{t_1/t_0} and Paasche P_{t_1/t_0} price indexes for the quantities and prices set in

[3] Note that the sum of α^i equals 1

previous example between year 1 and year 2, with year 1 as the baseline period.

$$
\begin{aligned}
L_{t_1/t_0} &= \frac{p_{t_1}^{squid} * q_{t_0}^{squid} + p_{t_1}^{sweet-pepper} * q_{t_0}^{sweet-pepper}}{p_{t_0}^{squid} * q_{t_0}^{squid} + p_{t_0}^{sweet-pepper} * q_{t_0}^{sweet-pepper}} * 100 \\
L_{t_1/t_0} &= \frac{7*90+2*270}{4*90+1*270} * 100 \\
L_{t_1/t_0} &= 185.7
\end{aligned}
$$

$$
\begin{aligned}
P_{t_1/t_0} &= \frac{p_{t_1}^{squid} * q_{t_1}^{squid} + p_{t_1}^{sweet-pepper} * q_{t_1}^{sweet-pepper}}{p_{t_0}^{squid} * q_{t_1}^{squid} + p_{t_0}^{sweet-pepper} * q_{t_1}^{sweet-pepper}} * 100 \\
P_{t_1/t_0} &= \frac{7*60+2*240}{4*60+1*240} * 100 \\
P_{t_1/t_0} &= 187.5
\end{aligned}
$$

We can also use the fact that the Laspeyres index and Paasche index are respectively the weighted arithmetic and the harmonic mean of the elementary indexes. Here we give the example for the Laspeyres index. First, we calculate the α_i.

Squid	Sweet pepper
$\frac{4*90}{4*90+1*270} = 0.57$	$\frac{1*270}{4*90+1*270} = 0.43$

Table 5: Weights of goods in total consumption

We now use the elementary indexes computed above and the weights.

$$
\begin{aligned}
L_{t_1/t_0} &= 0.57 * 175 + 0.43 * 200 \\
&= 185.7 \quad \text{which is the result found above}
\end{aligned}
$$

Properties of a Price Index For building a price index, such as the CPI (Consumer Price Index), one needs 4 things :

- A basket of goods and services : a set of multiple items depicting the consumption structure. In our Tataouine example, we choose for the sake of simplicity only two items but the French Consumer Price Index uses a sample of 114,000 items.
- a reference population : which would be used for setting weights. In the Euro Zone, the weights are set according to National Account.
- a baseline period (usually a year).
- a monitoring of prices

Fallacious graphical representations A index is measuring changes not values. Considering the graph below (Figure 1), one cannot conclude that water is more expensive than whisky since 2006 but one can rather conclude that the price of mineral water is increasing faster than the price of whisky.

If a price index in base 100 is bigger than 100, it means that with respect to the baseline period, prices have increased. Nevertheless, if an index is decreasing between two periods although staying higher than 100 (for instance going from 123 in year 2 to 112 in year 3), prices are indeed decreasing while still higher than in baseline period.

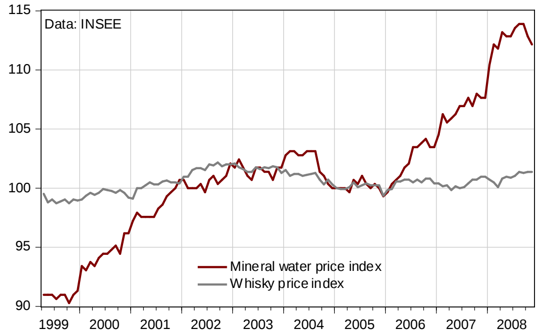

Figure 1: Whisky and Mineral Water Price indexes

There is no perfect index as you see in previous examples, different indexes do not give the same results, and thus do not give the same image of reality. Moreover, indexes are more difficult to build when prices, quantities, and the nature of good change. For instance, between 1900 and 2000, can one compare the price of vehicles - horse carts vs. cars - and fuels - oat vs. gasoline - ? Indeed, index number are *growing old*. Laspeyres indexes, in particular, are facing this puzzle due to the fact that they fixed prices and quantities at a baseline period. However, in practice, statistical institute are changing weights every year and build *chain indexes*, but that is beyond the scope of this course.

2 Use of price indexes and other indexes

Purchasing power index A purchasing power index can be built by dividing an index of wages by an index of prices.
Formally :

$$\text{Index of purchasing power of wages} = \frac{\text{Index of wages}}{\text{Index of prices}}$$

Terms of trade Terms of trade (TOT) are the relative price of imports in terms of imports. That is :

$$\text{Terms of Trade} = \frac{\text{Price of exports}}{\text{Price of imports}}$$

One can also build an index of Terms of Trade as the ratio between index of export's price and an index of import's price.

$$\text{Index of Terms of Trade} = \frac{\text{Index of export prices}}{\text{Index of import prices}}$$

If the index of terms of trade is inferior to 1 (or 100, depending on the chosen base) then it means that the price of exports grows slower (decrease faster) than price of imports grows (decrease), we say that **terms of trade are deteriorating**. If the index of Terms of Trade is bigger than 1, we say that **terms of trade are improving**.

Inflation and purchase power parity

Definition 2.1. The Law of One Price
In the absence of transaction costs a good must be sold for the same price in all locations. It means that there is no **arbitrage**. Arbitrage is the situation when one can take advantage of price differences for the same item in different markets.

Definition 2.2. Purchasing power parity
The theory that states that a unit of any given currency should be able to buy the same quantity of goods in all countries[4].

[4]This statement is false in practice due to transaction costs and to differences of productivity between countries, see the Big Mac Index

According to the Purchasing Power Parity, if in year 1, two currencies a and b are such that a quantity x of a equals a quantity y of b (formally : $x_a = y_b$), and that π_a is the level of inflation is the country A with the currency a, π_b the level of inflation in the country B with the currency b, then we should have :

$$x_a * (1 + \pi_a) = y_b * (1 + \pi_b)$$
$$x_a = y_b * \frac{(1 + \pi_b)}{(1 + \pi_a)}$$
$$\rightarrow \text{the currency } a \text{ should depreciate/appreciate by } \frac{(1 + \pi_b)}{(1 + \pi_a)} - 1$$

3 Elasticity

Imagine that some event drives up the price of gas in France. It could be a war in Russia that disrupts the world supply of gas, a booming Indian economy that boosts the world demand for gas. How would French consumers respond to the higher price ?
In broad fashion : Consumers would buy less. [5]
But you might want a precise answer : by how much would consumption of gas fall. This question can be answered using a concept called **elasticity**

Definition 3.1. Elasticity A measure of the responsiveness of a quantity demanded or supplied to a change in one of its determinant.
Formally

$$\text{Elasticity} = \frac{\frac{\Delta Y}{Y}}{\frac{\Delta X}{X}}$$
$$\rightarrow \text{How does a change in X (cause) affect Y (result)}$$

Usually, we use the concept for "price elasticity of demand" and we ask : By how much a change in the demand of a good responds to a change in the price of that good, computed as the percentage change in quantity demanded divided by the percentage change in price.

3.1 Applications

Elastic or inelastic good A good is said to be elastic if the quantity demanded responds substantially to a change in price. If it does not, the good is said to be inelastic. Thus, staple products like rice or bread are inelastic goods whereas Louboutin shoes and Picasso paintings are elastic goods. Usually, necessities have inelastic demand whereas luxuries have elastic demand.

Substitutes & Complements If the demand for good 1 increases with the price of good 2, then the goods are said to be substitutes. In the contrary, if the demand for good 1 decreases when the price of good 1 increases, then the goods are said to be complements. Thus, butter and margarine are substitutes whereas butter and honey are complements.
Formally, we talk about cross elasticity of demand and we compute :

$$\frac{\frac{\Delta Q_1}{Q_1}}{\frac{\Delta P_2}{P_2}} \begin{cases} > 0 & \text{goods are substitutes} \\ < 0 & \text{goods are complements} \end{cases}$$

Giffen good Usually, demand for a good decreases when its price increases (this is the law of demand). But there is a certain kind of goods for which this is not the case : basic necessities. When the demand of a good increases when its prices increases, we call it a **Giffen good**. The economic mechanism is as follows : if the price of rice rises in a poor countryside of China, people are likely to decrease their consumption of "luxuries" (like meat) to afford to buy more rice. And they will buy more to offset the meat deprivation. Thus, demand for rice is likely to increase.

There is another kind of goods for which demand increases with their price : luxury goods. They are called **Veblen goods**. The high price of that kind of good will be a sign of buyer's high social-status by way of

[5] adapted example from Greg Mankiw, Principles of Macroeconomics

conspicuous consumption. Thus, for Veblen goods, demand increases with their price.

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