Quantitative Tools - Level 1 - Fall 2015
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Lecture 10

Interest rates

Introduction & Definitions

An interest is the payment of a loan. Historically, it seems that the first loans where loan of cattle or food. It was then *fair* to charge interest since the cattle reproduce themselves and as do seeds. We could think about it as a cost of the loss of the opportunity, indeed the money-lender loose the opportunity to do something else. There are basically two different kind of interest :

- **simple interest** when the interest are paid at the end of the contract. The interest are calculated *pro rata temporis* - in proportion of the loan's duration.
- **compound interest** when the interest are added to the principal so that the interest earns interest from then on.

In this chapter, we will consider that a year has 360 days, divided in twelve months of 30 days each. We use the following notations : Let A_0 be the amount of money invested at the annual interest i, let A_n be the amount of money after n years.

1 Simple Interest

For simple interest, the interest are not added up to the principal.

1.1 Basic formulas - Principle

- After one year we have

$$A_1 = A_0 * (1 + i)$$

- After n years we have

$$A_n = A_0 * (1 + n * i)$$

- Let A_x be the amount of money invested for x months at the annual interest rate i. Then

$$A_x = (1 + \frac{x * i}{12})A_0$$

- Let A_d be the amount of money invested for d days at the annual interest rate i. Then

$$A_d = A_0 * (1 + \frac{d * i}{360})$$

1.2 Use of Basic formulas

Let A_0 be the amount of money invested for d days at the annual interest i. In this subsection, we will derive from the previous formula, formula for the interest rate, the length of the investment, for the present value of the investment, while knowing the others parameters of the problem. Indeed, the problem has four parameters : the amount of money invested, the annual interest rate, the length of the investment and amount of the interests at the end of the investment. Knowing three parameters, we are able to find the fourth.

- If we are looking for the present value of A_d, the money we'll get at the end of the d days, that is expressing A_0 as a function of A_d. We have

$$A_0 = \frac{A_d}{1 + \frac{d * i}{360}}$$

- If we look for the interest rate :

$$i = \frac{A_d - A_o}{A_0} * \frac{360}{d}$$

- If we look for the length d of the investment :

$$d = \frac{A_d - A_0}{A_0} * \frac{360}{i}$$

1.3 Interest checked off - effective rates of investment

When the interest are paid at the beginning of the investment, we used the term *Interest checked off.* Because of this payment, the money investment is not the same. Indeed, we *really* invest only the principal minus the interest checked off. As a consequence, the effective rate of investment has changed.
Let A_0 the capital invest at the beginning, at the annual interest i, during n days, with checked off interest rates.
The amount of money which is indeed invested is :

$$A_0 - \text{the interest paid} = A_0 * (1 - \frac{i * d}{360})$$

The interest rate have also changed, and applying the formula found above, we get a new effective rate of investment i':

$$i' = \frac{A_0 - A_0 * (1 - \frac{i * d}{360})}{A_0 * (1 - \frac{i * d}{360})}$$

$$i' = \frac{i}{1 - \frac{i * d}{360}}$$

1.4 Discount

A discount is a deduction from the usual cost of something, typically given for prompt or advance payment. In finance, it is a percentage deducted from the face value of a bill of exchange or promissory note when the payment is done before the due date. With interest rates, the *timing* matters. Receiveing one hundred euros today allows you to go to the bank, put it on a saving accounts and receive the interest. Receving it in one month is a *loss* of value.

Definition 1.1. Present Value
The present value, PV_A of a amount of money A that will be received (or due) in d days is the amount of money that should be invested now at the interest rate i so that the principal and the interests equals A in d days. We call i the discount rate
Formally,

$$PV_A = \frac{A}{1 + \frac{d * i}{360}}$$

The present value is used by firms for comparing investments. Indeed, there is three things to be compared : the amount of money invested, the returns and the *timing of events.* Thus, in order to make things *comparable* one need to transform all the future flows of money in *present value terms.* The discount rate chosen is usually the rate of a riskless bond (like the US Treasury bonds).

There are two kinds of discount :

- the rational discount : the difference between an amount of money and its present value.
- the commercial discount : the value of the interests received if it the amount of money is invested at an interest rate equal to discount rate.

Definition 1.2. Rational Discount
Let A_d be the amount of money due in d days. Let i be the discount rate. The rational discount is the difference between A_d and its present value.
That is :

$$\begin{aligned} Rational\ Discount &= A_d - \frac{A_d}{1 + \frac{d*i}{360}} \\ &= A_d\left(1 - \frac{1}{1 + \frac{d*i}{360}}\right) \end{aligned}$$

Definition 1.3. Commercial Discount
Let A_d be the amount of money due in d days. Let i be the discount rate. The commercial discount is :

$$Commercial\ Discount = A_d * \frac{d*i}{360}$$

Definition 1.4. Commercial Present Value
With the *commercial discount* we can define a **commercial present value**, equals to the difference between the money due in d days and the commercial present value.

$$\begin{aligned} Commercial\ Present\ Value &= A_d - A_d * \frac{d*i}{360} \\ &= A_d\left(1 - \frac{d*i}{360}\right) \end{aligned}$$

Definition 1.5. Equivalence of debts contracts
Two debts contracts are equivalent at time t if the commercial present value of the two debts contracts is the same. Date t is **the equivalence date**.
Formally, let A_d and $A_{d'}$ be two debts contracts due in d and d' days respectively. Let i be the discount rate. We have :

$$\begin{gathered} \text{Commercial Present Value of } A_d = \text{Commercial Present value of } A_{d'} \\ A_d(1 - \frac{d*i}{360}) = A_{d'}(1 - \frac{d'*i}{360}) \end{gathered}$$

From that formula, one can deduce $A_{d'}$:

$$A_{d'} = A_d \frac{(1 - \frac{d*i}{360})}{(1 - \frac{d'*i}{360})}$$

One can note that $A_{d'}$ is increasing with d' : the longer the maturity, the higher the interests, which is quite intuitive.

Example 1.1. Extension of maturity
A firm has a repayment due in 30 of 50,000$ and want to renegotiate the debt contract so that she will pay an amount A_{60} in sixty days. The commercial discount rate is of 5%. We want to find the amount A_{60} due in 60 days.

$$\begin{aligned} 50,000 * (1 - \frac{30*5}{360*100}) &= A_{60} * (1 - \frac{60*5}{360*100}) \\ A_{60} &= 50,000 * \frac{(1 - \frac{30*5}{360*100})}{(1 - \frac{60*5}{360*100})} \\ A_{60} &= 50,210 \end{aligned}$$

Definition 1.6. Average maturity
Let $A_{d_1}, ..., A_{d_n}$, be n due payments in $d_1, ..., d_n$ days respectively, and i the annual interest rate. Let A_{d^*} be the sum of all the A_{d_i}. The **average maturity** is the number of days d^* so that a single payment A_{d^*} made in d^* days is equivalent to the n payments made in $d_1, ..., d_n$ days respectively, that is, equals the n payments in commercial present value terms.
Formally :

$$A_{d^*}(1 - \frac{d^* * i}{360}) = \sum_{j=1}^{n} A_{d_j} * (1 - \frac{i * d_j}{360})$$
$$d^* = \frac{\sum_{j=1}^{n} A_{d_j} * d_j}{A_{d^*}}$$

2 Compound Interest

Definition 2.1. Compount Interest
Compound interests are interests calculated on the initial principal and on the accumulated interest of previous period. Compound interests are *interests on interests.*

2.1 Basic Formulas

Let A_0 be the amount of money invested, A_n the amount of money after n years, i the annual interest rate, and I the interest earned.

$$A_n = A_0 * (1 + i)^n$$

From that formula we can derive :

$$A_0 = \frac{A_n}{(1 + i)^n}$$
$$i = \left(\frac{A_n}{A_0}\right)^{1/n} - 1$$
$$n = \frac{ln(\frac{A_n}{A_0})}{ln(1 + i)}$$

$$I = A_n - A_0$$
$$= A_0((1 + i)^n - 1)$$

2.2 Equivalent Rates, Proportional Rates & Average Rates

Definition 2.2. Equivalent Rates
Let A_0 be a capital invested at the annual rate i. At the end of the year, one get $A_1 = A_0 * (1 + i)$.
If the same capital is invested during one year but interests are compounded p times at the interest rate i_p. At the end of year, we get $A_{1'} = A_0(1 + i_p)^p$. The two interest rates was **equivalent** iff $S_1 = S_{1'}$.
Formally,

$$S_1 = S_{1'}$$
$$1 + i = (1 + i_p)^p$$
$$i_p = (1 + i)^{1/p} - 1$$

Definition 2.3. Proportional Rates
The annual interest rate i is propotionnal to the interest i_p, compounded p times in a year if :

$$i_p = \frac{i}{p}$$

Definition 2.4. Average Rates
Let A_0 be a capital invested at time $t = 0$ for n years. Let be i_k the annual interest rate during year k. We have :

$$A_n = A_0 * (1 + i_1) * (1 + i_2) * ...(1 + i_n)$$
$$A_n = A_0 * \Pi_{k=1}^{n}(1 + i_k)$$

Let $\tilde{i}$ be the average interest rate. We use the formula for the average growth rate (see Lecture 1)

$$\tilde{i} = (\Pi_{k=1}^{n}(1 + i_k))^{(1/n)} - 1$$

2.3 Present Value with compound interests

Definition 2.5. Present Value in case of compound interests
Let A_n be the value in n years of a capital A_0 invested today at the annual interest rate i.

$$A_n = A_0(1 + i)^n$$
$$A_0 = \frac{A_n}{(1 + i)^n}$$

A_0 is the **present value** of A_n

Notice that it means that **the amount of a loan equals the present values of the future repayments.**

We can use this formula to derive in case of compound interests of what we've seen with simple interests, that is

- loan repayments
- replacement of several loans by a single loan
- average maturity

The following examples go into each case.

Example 2.1. Loan repayments
Bob make a loan of 100, 000$ the 1st of January of 2001 at an interest rate of 6% for three years. Bob make a first repayment at the end of the first year of 20, 000$ and a second repayment at the end of the second year of 40, 000$. How much will Bob will have to repay at the end of the third year ?
There are basically two ways to answer that question :

- compute for each year, the interest due and the remaining principal
- write the equivalence of the present values.

The **first method** leads to :

Date	Due Capital	Due Interests	Repayments	Capital due after repayments
01/01/2001	100,000			
01/01/2002	100,000	6,000	20,000	86,000
01/01/2003	86,000	5,160	40,000	51,160
31/12/2003	51,160	3,069.60	X	

We finally compute $X = 51,160 + 3,069.60 = 54,229.50\$$

Using the **second method**, we get :

$$100,000 = \frac{20,000}{(1+i)^1} + \frac{40,000}{(1+i)^2} + \frac{X}{(1+i)^3}$$
$$X = (1.06)^3 * \left(100,000 - \frac{20,000}{1.06} - \frac{40,000}{1.06^2}\right)$$
$$X = 54,229.60\$$$

Example 2.2. Replacement of several loans by a single loan Alice has three loans with different due capital and maturiy

- 15,000$ due in one year
- 40,000$ due in two years
- 55,000$ due in three years.

All the loans have a annual interest of 5.4%. Alice wants to make a single repayment in five years. How much will she need to repay in five years ?
Let call X this amount of money. We write down the *equivalence of present values* :

$$\frac{X}{(1+i)^5} = \frac{15,000}{(1+i)^1} + \frac{40,000}{(1+i)^2} + \frac{55,000}{(1+i)^3}$$
$$X = 126,448.61\$$$

Example 2.3. Average maturity
We use the same data than in the previous example. But Alice now wants to make a single repayment of the three loans of 110,000\$. That is, Alice wants to know the *average maturity* of its loans. Let n be that average maturity. We write down the *equivalence of present values* :

$$\frac{110,000}{1.054^n} = \frac{15,000}{(1+i)^1} + \frac{40,000}{(1+i)^2} + \frac{55,000}{(1+i)^3}$$
$$1.054^n = \frac{110,000}{97,210.02}$$
$$1.054^n = 1.13157$$
$$n = \frac{ln 1.13157}{ln 1.054}$$
$$n = 2.35 \text{ years}$$

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